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Accurate formula for determination of natural frequencies of FGM plates basing on frequencies of isotropic plates

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Abstract

The functionally graded materials (FGM) have many advantages especially in thermal environments, and in the recent years they have been used in many engineering applications. A FGM plate is an inhomogeneous composite made of two constituents (usually ceramic and metal), with both the composition and the material properties varying smoothly through the thickness of the plate. The FGM plate vibrations have a strong bending-stretching coupling effect. Large computational efforts are required for calculating by numerical methods the natural frequencies of such plates with different volume fractions of containing materials. Basing on the investigation of tendencies in frequency variation due to different volume fraction, the present study derives an empiric accurate formula that gives a correlation between the frequencies of FGM plate and isotropic ones made of containing materials, even with different Poisson ratio. The formula gives immediately accurate results for different vibrational modes and for various volume fractions of containing materials without expending much computational effort. Numerical example is presented in order to demonstrate the accuracy, applicability and convenience of using the suggested formula. The natural frequencies calculated with the presented formula are compared with those obtained through other numerical method for thick FGM annular plates.

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1. Introduction

A functionally graded material (FGM) plate is an inhomogeneous composite made of two constituents (usually ceramic and metal), with both the composition and the material properties varying smoothly

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through the thickness of the plate. The FGM plate vibrations have a strong bending-stretching coupling effect. Large computational efforts are required for calculating by numerical methods the natural frequencies of such plates with different volume fractions of containing materials.

Basing on examination of published results for transverse vibrations of rectangular plates Abrate [1] derived proportionality constants between the natural frequencies of the FGM plates and those of the isotropic plates using the ratios of previously calculated frequencies of isotropic ones. In the same manuscript noted that completely ignoring the extension-bending coupling phenomenon the proportionality constants are proportional to $(D_{11}/m)^{1/2}$ and then can be approximately predicted. Here D_{11} is bending stiffness coefficient of moment-curvature relation and m denote the mass per unit area. Another improved approximation was given by the same author [2] by eliminating extension-bending coupling using the different reference surface instead the mid-plane in which the transverse motion is uncoupled from the in-plane displacement. It should be noted, that in this case the boundary conditions must be subscribed on the new reference surface, otherwise the coupling effect will be remained.

The aim of present paper is investigation of proportionality of the natural frequencies considering types of vibration modes. For this purpose investigation of free vibrations of FGM annular plates was done. The study of natural vibrations of isotropic and FGM annular plates [3] reveals that though the strong coupling effect in vibrational modes due to variation of density and elasticity in FGM plates, clearly expressed in-plane shear, mainly bending and mainly stretching modes occurred as in isotropic annular plates. This fact allows investigating the correlation of natural frequencies of different types of vibrational modes for various cases of constituent materials.

Basing on the investigation of tendencies in frequency variation due to different volume fraction, the present study derives an empiric but accurate formula that gives a correlation between the frequencies of FGM plate and isotropic ones made of separate containing materials. The formula gives immediately accurate results for different vibrational modes and for various volume fractions of containing materials with different Poisson ratio without expending much computational effort.

Numerical example is presented in order to demonstrate the accuracy, applicability and convenience of using the suggested formula. The natural frequencies calculated with the presented formula are compared with those obtained through other numerical methods for thick FGM annular plates.

2. Variation of material properties in FGM plates

FGM can be fabricated by continuously intermixing several materials without generating a boundary. The properties change gradually with position. The property gradient in the material is caused by a position-dependent chemical composition and can be defined by the so-called transition function, which describes the concentration of the component as a function of position.

For functionally graded materials with two constituent materials the variations through the thickness of material properties P can be expressed as

$$P(\bar{z}) = (P_t - P_b)V_t(\bar{z}) + P_b \tag{1}$$

Here P can represent Young's modulus E, Poisson ratio μ , and the mass density ρ , and $V_t(\bar{z})$ is the volume fraction variation of the top material, and it is assumed to follow a power-law distribution [4] as

$$V_t(\bar{z}) = \left(\bar{z} + \frac{1}{2}\right)^g , \tag{2}$$

where $-1/2 \le \bar{z} \le 1/2$ is non-dimensional coordinate through the thickness from the middle surface topward, and g is a gradient index. The volume fractions V_t of the top and V_b of the bottom constituent materials in the entire cross-section are obtained by integration of volume fraction function trough thickness as follow

$$V_t = \int_{-1/2}^{1/2} \left(\bar{z} + \frac{1}{2} \right)^g d\,\bar{z} = \frac{1}{1+g} \quad ; \quad V_b = 1 - V_t = 1 - \frac{1}{1+g} = \frac{g}{1+g} \tag{3}$$

3. Correlations in natural frequencies of isotropic and FGM annular plates

The extensive study of natural vibrations of isotropic plates and shells that have been performed by many researchers shows that natural frequencies of all isotropic plates with the same external sizes ratios, Poisson ratio and boundary conditions can be represented by dimensionless frequency parameter, for instance as

$$\Omega = \omega a \sqrt{\rho / E}\ , \tag{4}$$

where a can be one of plate dimension like thickness, side size of rectangular plate or radius of circular or annular plate. In this form the frequency parameter can represent the values obtained by analysis basing on all of two- and three-dimensional plate theories. This dimensionless frequency parameter allows calculate the circular frequency ω of the plate made of any material with any given specific characteristics. It also gives the correlation between natural frequencies in same vibrational mode of the plates made from different isotropic materials with equal Poisson ratio, and allows easily calculate the vibrational characteristics for isotropic plates with any material properties basing on previously obtained ones for the same geometrical configuration as

$$\omega_{\mathrm{mat2}} = \omega_{\mathrm{mat1}} \sqrt{(\rho_1 E_2)/(\rho_2 E_1)} \tag{5}$$

In case of two isotropic plates made of materials with different Poisson ratio this exact correlation unfeasible and separate analysis for each plate is required.

In Table 1 the values of natural frequency parameters for thick annular plates obtained basing on the first-order shear deformation shell theory and using Dynamic Stiffness method and Exact Element method [3] are presented. The plates are fully free in inner and outer boundaries. The values given for two plates made of isotropic materials with significant difference in Poisson ratios: silicone-nitride Si_3N_4 with μ=0.24, E=322271.5 MPa, ρ=2370 kg/m^3, and stainless steel SUS304 with μ=0.31776, E=207787.7 MPa, ρ=8166 kg/m^3. For each natural frequency the notation of type of corresponding vibration mode is given (M.S.-Meridianal stretching mode; B- Bending mode; I.P.S.-In-plane shear mode; R.S.-Radial stretching mode) and besides, ratios between the frequency parameters of corresponding modes of such plates are calculated.

From the presented results easy to recognize that the correlation factors for identical type of vibrational modes are same, but various for different types of modes. For example, for plate described in mentioned above case the correlation factors are: approximately 0.99 for bending modes, approximately 0.96 for meridianal stretching modes and exactly 1.03088 for all torsional modes due to in-plane shear. The ratio 1.03088 exactly represents the value of $\sqrt{(1+\mu_1)/(1+\mu_2)}$. The difference between correlation ratios reaches up to 7.5 percents.

Table 1. Comparison of nondimensional frequencies Ω of completely free annular plates (R_{inner}/R_{outer}=0.1, H/R_{outer}=0.1)

Appearance sequence number	Isotropic material 1 (μ=0.31776)		Isotropic material 2 (μ=0.24)			FG material (g=1)		
	Mode type	frequency parameter Ω_{mat1}	Mode type	Frequency parameter Ω_{mat2}	$\frac{\Omega_{mat2}}{\Omega_{mat1}}$	Mode type	Frequency parameter Ω_{FGM}	$\frac{\Omega_{FGM}}{\Omega_{mat1}}$
1	M.S.	8.6728	M.S.	8.3655	0.96457	M.S.	11.8849	1.37036
2	B.	35.9747	B.	35.3451	0.98250	B.	49.7560	1.38308
3	M.S.	68.7453	M.S.	65.9001	0.95861	M.S.	94.4683	1.37418
4	B.	78.1304	B.	77.1411	0.98734	B.	108.3952	1.38736
5	I.P.S.	104.0416	I.P.S.	107.2540	1.03088	I.P.S.	148.6484	1.42874
6	B.	131.4750	B.	130.2180	0.99044	B.	182.7148	1.38973
7	**I.P.S.**	**171.1129**	R.S.	171.0782	0.98531	R.S.	242.1093	1.39441
8	R.S.	173.6283	**I.P.S.**	**176.3962**	**1.03088**	**I.P.S.**	**244.3699**	**1.42812**
9	B.	192.1120	B.	190.8440	0.99340	B.	267.5628	1.39274
10	I.P.S.	237.4979	I.P.S.	244.8310	1.03088	I.P.S.	338.9597	1.42721

Another important and clearly observed phenomenon is interchanging in appearance sequence of same vibrational modes due to variation in Poisson ratio. For example, the 7-th and 8-th frequencies of isotropic plates with different Poisson ratio have different types of vibrational modes. Owing to it, the same phenomenon will occur in case of FGM plates with different volume fractions of constituent materials if they have different Poisson ratio.

Beside results for isotropic plates In Table 1 presented the first ten natural frequency parameters for thick FGM annular plate with equal volume fractions (g=1) of two constituent materials- silicone-nitride Si_3N_4 and stainless steel SUS304. The upper surface made from silicone-nitride. The nondimensional frequency parameter for FGM plates can be defined with correlation to material properties of either bottom or top constituent material. In this study it defined as

$$\Omega_{FGM} = \omega_{FGM} a \sqrt{\rho_b / E_b} \tag{6}$$

Comparison of frequencies with those of isotropic plates shows the same above mentioned tendencies. In case of in-plane shear modes the correlation factor ceases to be constant, and slightly changes in vicinities of value 1.428 due to effect of rotary inertia. In distinction from isotropic steel plate for which the 3-rd in-plane shear mode occurs at seventh frequency, in case of FGM plate with g=1 the in-plane shear mode appear at the next sequence number. The phenomenon of appearance interchanging between modes of FGM plates with different volume fraction can be observed in Table 3.

All this leads to conclusion that in aim of achievement of more exact prediction of natural frequencies of FGM plates the correlation between frequencies of FGM plates with those of isotropic plates should be obtained separately for each type of vibrational modes and basing on frequencies both of two isotropic plates made of constituent materials.

4. The approximate formula.

Basing on the statement that functionally graded plates behave like homogeneous plates [2] we proposed an approach for correlation between natural frequencies of FGM and isotropic plates.

Let's define an isotropic material that equivalent to FGM by density and elasticity modulus with the following properties

$$E_{eq} = E_t V_t + E_b V_b \ , \quad \rho_{eq} = \rho_t V_t + \rho_b V_b \ , \tag{7}$$

where V_t and V_b as defined in Eq.(3). Then we can use Eq. (6) for obtaining the approximate frequency values of equivalent material plate Ω_{eq} from those of isotropic plate Ω_{is} made of one of the constituent isotropic materials, either top or bottom as follow:

$$\Omega_{eq} = \sqrt{(\rho_{is} E_{eq})/(\rho_{eq} E_{is})} \cdot \Omega_{is} \tag{8}$$

In case that two constituent materials have the same Poisson ratio, Eq.(8) will give the same values basing on either top or bottom materials due to proportional correlation between ω_t and ω_b described in Eq. (5). However, in most cases constituent materials of FGM have different Poisson ratio and this difference is remarkable as in case of SUS304 and Si3N4. In this case Eq. (8) will give two different results for each vibration mode basing on frequency parameter of different material plates.

In order to achieve correlation formula that will give singular value, the examination of these different values and their discrepancy to exact values was done. In Table 2 the comparison of approximated 2nd, 4th and 5th natural frequencies of equivalent material plate obtained by Eq. (8) with exact ones of FGM plate obtained by approach given in [3] for different values of gradient index and three types of vibrational modes are presented.

Table 2. Comparison of dimensionless frequency parameters of completely free annular plate (R_{inner}/R_{outer}=0.5, H/R_{outer}=0.1) obtained by exact numerical analysis with those obtained by Eq. (8) basing on frequencies of isotropic plates.

Mode type		M.S.	I.P.S.	B.	M.S.	I.P.S.	B.
FG material	Gradient index g (volume fractions)	g=5 (V_t=0.833, V_b=0.167)			g=0.2 (V_t=0.167, V_b=0.833)		
	Ω_{eq} basing on Ω_{is} for Si_3N_4	50.4563	158.1465	204.0038	85.6919	268.5864	346.4676
	Exact Ω_{FGM} by approach given in [3]	50.7221	154.2903	210.6648	85.7800	267.2446	342.9480
	Ω_{eq} basing on Ω_{is} for SUS304	50.7738	153.4098	206.2162	86.2311	260.5418	350.2249
Isotropic material	Ω_{is} of Si_3N_4 plate (μ=0.31776)	45.3419	79.7997	142.1164			
	Ω_{is} of SUS304 plate (μ=0.24)	45.6272	81.1618	137.8598			

A certain general trend in position of approximate values in relation to the exact can be observed. As can be seen from Table 2, the approximate frequency parameters of equivalent plate that obtained basing on exact frequency values of isotropic plate made from material that have dominant volume fraction in FGM is closer to the exact frequency value of FGM plate than the another one. Considering this tendency, a single value of frequency parameter for FGM plate can be found by using the volume fractions of constituent materials as weighting factors for two approximate frequency values for the corresponding vibrational mode obtained basing on two isotropic plates frequencies. This approach can be formulated by following approximation formula

$$\Omega_{FGM}^{approx} = \sqrt{\frac{\rho_b E_{eq}}{\rho_{eq} E_b}} \left(\Omega_b \cdot V_b + \Omega_t \cdot V_t \right) \quad \text{or} \quad \omega_{FGM}^{approx} = \omega_b \sqrt{\frac{\rho_b E_{eq}}{\rho_{eq} E_b}} \cdot V_b + \omega_t \sqrt{\frac{\rho_t E_{eq}}{\rho_{eq} E_t}} \cdot V_t \ , \tag{9}$$

where Ω_b, ω_b, Ω_t and ω_t are dimensionless natural frequency parameters and absolute natural frequencies of isotropic plates with Poisson ratios as of bottom and top constituent materials respectively.

In order to demonstrate the accuracy, applicability and convenience of using the suggested approximate formula, approximate values of natural frequency parameters were calculated for thick annular FGM plates with constant thickness and completely free boundary conditions and presented in Table 3. Two different gradient indexes are considered: g=1 and g=5. The obtained results are compared with those obtained by numerical analysis using approach given in [3].

Table 3. Comparison of dimensionless frequency parameters Ω_{FGM} of completely free annular plate (R_{inner}/R_{outer}=0.1, H/R_{outer}=0.1) obtained by exact numerical analysis with those obtained by approximation formula basing on frequencies of two isotropic plates.

Appearance sequence number	FGM plate g=1.0				FGM plate g=5.0			
	Mode type	Numerical analysis [3]	Approximation formula Eq.(9)	Error %	Mode type	Numerical analysis [3]	Approximation formula Eq.(9)	Error %
1	M.S.	11.8849	11.9789	0.79	M.S.	9.7839	9.5941	-1.94
2	B.	49.7560	50.1417	0.78	B.	40.7922	39.9157	-2.15
3	M.S.	94.4683	94.6630	0.21	M.S.	75.8543	75.9717	0.15
4	B.	108.3952	109.1644	0.71	B.	88.6785	86.7597	-2.16
5	I.P.S.	148.6484	148.5524	-0.06	I.P.S.	116.4568	116.3728	-0.07
6	B.	182.7148	183.9844	0.69	B.	149.1600	146.0716	-2.07
7	R.S.	242.1093	242.3475	0.10	**I.P.S.**	**191.4752**	**191.3936**	**-0.04**
8	**I.P.S.**	**244.3699**	**244.3179**	**-0.02**	R.S.	192.6139	192.7399	0.07
9	B.	267.5628	269.2390	0.63	B.	217.8565	213.5463	-1.98
10	I.P.S.	338.9597	339.1036	0.04	I.P.S.	265.6467	265.6467	0.00

The approximate frequency values are very close to those obtained by numerical analysis. The difference between the approximated values and exact ones is less than 2.5% for transverse vibration modes, and for in-plane shear modes and radial stretching modes is always less than 0.1%.

5. Conclusions

The study of correlations of natural frequencies of FGM plates shows that their prediction should be performed considering types of vibrational modes. The suggested approximate formula for predicting natural frequencies of such plates gives immediately accurate results for different vibrational modes and for various volume fractions of containing materials without expending much computational effort.

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